

# PROOF OF THE KONTSEVICH NON-COMMUTATIVE CLUSTER POSITIVITY CONJECTURE

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**ABSTRACT.** We extend the Lee-Schiffler Dyck path model to give a proof of the Kontsevich non-commutative cluster positivity conjecture with unequal parameters.

Let  $k$  be any field of characteristic zero. For any  $r \in \mathbb{Z}_{>0}$ , consider the following  $k$ -linear automorphism of the skew-field  $K = k(x, y)$  of rational functions in non-commutative variables  $x$  and  $y$ :

$$F_r : (x, y) \mapsto (xyx^{-1}, (1 + y^r)x^{-1}).$$

Our main result is the following.

**Theorem 1** (Kontsevich conjecture). *For any  $r_1, r_2 \in \mathbb{Z}_{>0}$  and any  $k \geq 0$ , the elements  $x_k = \underbrace{F_{r_1} F_{r_2} F_{r_1} \cdots}_k(x)$  are given by non-commutative Laurent polynomials in  $x$  and  $y$  with non-negative integer coefficients.*

**Remark 2.** *Using a symmetry argument, Theorem 1 implies an analogous statement for  $y_k = \underbrace{F_{r_1} F_{r_2} F_{r_1} \cdots}_k(y)$ .*

The Laurentness of these expressions was established by Usnich [4] for  $r_1 = r_2$  and by Berenstein-Retakh [1] for general  $r_1, r_2$ . The positivity was shown by Di Francesco-Kedem [2] for  $r_1 r_2 = 4$  and by Lee-Schiffler [3] for  $r_1 = r_2$ . We follow the Lee-Schiffler approach in this note.

Fix integers  $r_1, r_2 \in \mathbb{Z}_{>0}$ . Our proof will make use of two-parameter Chebyshev polynomials  $U_{k,j}$ ,  $k, j \in \mathbb{Z}$ , defined recursively by:  $U_{-1,j} = 0$ ,  $U_{0,j} = 1$ ,  $U_{k+1,j+1} = r_j U_{k,j} - U_{k-1,j-1}$ , where  $r_j = \begin{cases} r_1, & \text{if } j \text{ is odd;} \\ r_2, & \text{if } j \text{ is even.} \end{cases}$

From now on we will work under the assumption  $r_1 r_2 \geq 5$ . The cases  $r_1 r_2 \in \{1, 4\}$  were settled in [4] and [2] and the remaining cases  $r_1 r_2 \in \{2, 3\}$  are given explicitly at [http://pages.uoregon.edu/drupel/dyck\\_examples.pdf](http://pages.uoregon.edu/drupel/dyck_examples.pdf).

Fix  $n \geq 2$ . Consider the rectangle  $R_n \subset \mathbb{Z}^2$  with corner vertices  $(0, 0)$  and  $(U_{n-3,1} - U_{n-4,2}, U_{n-4,2})$ . When  $R_n$  lies in the first quadrant, a *Dyck path* is a lattice path in  $R_n$  starting at  $(0, 0)$  and taking North or East steps to end at  $(U_{n-3,1} - U_{n-4,2}, U_{n-4,2})$  such that the path never crosses the main diagonal of  $R_n$  and the slope of each subpath beginning at  $(0, 0)$  does not exceed the slope of the main diagonal. Here we consider a vertical edge to have slope  $\infty$ . We modify this definition slightly when  $R_n$  lies in the second quadrant by replacing the East step with a diagonal  $(-1, 1)$ -upstep and considering vertical edges to have slope  $-\infty$ . When  $n = 2$ ,  $R_n$  lies in the fourth quadrant and we use a diagonal  $(1, -1)$ -downstep. We will call a Dyck path *maximal* if no subpath of another Dyck path lies closer to the main diagonal. Write  $D_n$  for the maximal Dyck path in  $R_n$ . The next Lemma follows by induction from the definitions.

**Lemma 3.** *Denote  $\epsilon_k := \max\{0, 2 - r_{k-1}\}$ ,  $\delta_k := \epsilon_k + 2\epsilon_{k-1} + 1$  for  $k \in \mathbb{Z}$ . Suppose  $k - \delta_k \geq 4$ . Then the Dyck path  $D_k$  consists of  $r_{k-\epsilon_{k-1}} - \delta_k + 1$  copies of  $D_{k-1-\epsilon_{k-1}}$  followed by a copy of  $D_{k-1-\epsilon_{k-1}}$  with its first  $D_{k-1-\delta_k}$  removed.*

Let  $U_n = \max\{|U_{n-3,1}|, |U_{n-4,2}|\}$  be the number of edges in  $D_n = (\omega_0, \alpha_1, \omega_1, \alpha_2, \dots, \alpha_{U_n}, \omega_{U_n})$ , where the vertices of  $D_n$  are labeled by  $\omega_0, \omega_1, \dots, \omega_{U_n}$  and  $\alpha_i$  is the edge connecting  $\omega_{i-1}$  and  $\omega_i$ . Let  $i_1, \dots, i_{U_{n-4,2}}$  denote the increasing sequence so that  $\alpha_{i_j}$  makes an upward step. We will write  $\nu_0, \dots, \nu_{U_{n-4,2}}$  for the sequence of vertices satisfying  $\nu_0 = (0, 0)$  and  $\nu_j = \omega_{i_j}$ .

**Definition 4.** For  $i < j$  denote by  $s_{ij}$  the slope of the line from  $\nu_i$  to  $\nu_j$  and by  $s$  the slope of the main diagonal of  $R_n$ . For  $0 \leq i < k \leq U_{n-4,2}$  let  $\alpha(i, k)$  be the subpath of  $D_n$  from  $\nu_i$  to  $\nu_k$  labeled/colored as follows:

- (1) If  $s_{it} \leq s_n$  for all  $t$  with  $i < t \leq k$ , then  $\alpha(i, k)$  is called a Dyck prefix (blue).
- (2) If  $s_{it} > s_n$  for some  $t$  with  $i < t \leq k$ , then
  - (a) if the smallest such  $t$  is of the form  $i + U_{m,2} - wU_{m-1-\epsilon_{m-1},2}$  for some integers  $1 \leq m \leq n-4$  and  $1 \leq w < r_{m-\epsilon_{m-1}} - \delta_m$ , then  $\alpha(i, k)$  is called an  $(m, w)$ -Dyck suffix (green).
  - (b) otherwise,  $\alpha(i, k)$  is called a short suffix (red).

Write  $\mathcal{P}(D_n) = \{\alpha(i, k) : 0 \leq i < k \leq U_{n-4,2}\} \cup \{\alpha_1, \dots, \alpha_{U_n}\}$  for the set of admissible subpaths of  $D_n$ . For  $\beta \subset \mathcal{P}(D_n)$  we define the support  $\text{supp}(\beta) \subset D_n$  in the natural way. We will use the term *hook* for the supports of the subpaths  $\alpha(k, k+1)$ . It will be convenient to refer to a hook as type 1, 2, or 3 depending on whether the horizontal displacement from the bottom to the top of the hook is  $r_2 - 1$ ,  $r_2 - 2$ , or  $r_2 - 3$ , respectively.

Call  $\beta \subset \mathcal{P}(D_n)$  an *overlapping* collection if there exists either  $\alpha(i, k), \alpha(i', k') \in \beta$  which share a vertex or  $\alpha_j, \alpha(i, k) \in \beta$  with  $\alpha_j \in \alpha(i, k)$ . We will need the following  $K$ -valued weightings on non-overlapping collections.

**Definition 5.** Write  $\varepsilon_i = \begin{cases} 1 & \text{if } \alpha_i \text{ is vertical;} \\ 0 & \text{otherwise.} \end{cases}$  For each non-overlapping collection  $\beta \subset \mathcal{P}(D_n)$  define


$$\beta_{[i]} = \begin{cases} y^{r_1-\varepsilon_i} x^{-1}, & \text{if } \alpha_i \notin \text{supp}(\beta); \\ y^{-\varepsilon_i} x^{-1}, & \text{if } \alpha_i \in \beta \text{ and } \alpha_i \text{ is not diagonal;} \\ x^1 y^{-1} x^{-1}, & \text{if } \alpha_i \in \beta \text{ and } \alpha_i \text{ is diagonal with an upstep;} \\ x^0 y^1, & \text{if } \alpha_i \in \beta \text{ and } \alpha_i \text{ is diagonal with a downstep;} \\ x^0 y^0, & \text{if } \alpha_i \in \alpha(j, k) \in \beta \text{ is horizontal;} \\ x^h y^{-1} x^{-1}, & \text{if } \alpha_i \in \alpha(j, k) \in \beta \text{ is the last edge of a hook of type } h. \end{cases}$$

We have the following refinement of Theorem 1.

**Theorem 6.** Suppose  $r_1, r_2 \in \mathbb{Z}_{>0}$ . Write  $q = xyx^{-1}y^{-1}$ . Then for  $n \geq 2$  we have  $x_{n-1} = \sum_{\beta \in \mathcal{F}(D_n)} q \prod_{i=1}^{U_n} \beta_{[i]}$ ,

where the product is taken in the natural order and the sum ranges over the set  $\mathcal{F}(D_n)$  of non-overlapping collections  $\beta \subset \mathcal{P}(D_n)$  subject to the conditions:

- C1: if  $\alpha_i$  is diagonal, then  $\alpha_i$  is supported on  $\beta$ ;
- C2: if  $\alpha(i, k) \in \beta$  is a short suffix, then the preceding non-diagonal edge of  $\nu_i$  is supported on  $\beta$ ;
- C3: if  $\alpha(i, k) \in \beta$  is an  $(m, w)$ -Dyck suffix, then at least one of the preceding  $U_{m-1,1} - wU_{m-2-\epsilon_{m-1},1}$  non-diagonal edges of  $\nu_i$  is supported on  $\beta$ .

**Example 7.** For  $r_1 = 2, r_2 = 3, n = 5$  we have  $U_{2,1} = 5, U_{1,2} = 2$  and so  $R_5$  and  $D_5$  are given by: .

We have the following expression for  $x_4$ :

$$x_4 = qxy^{-1}xy^{-1}x^{-1} + qxy^{-1}x^{-1}(1+y^2)x^{-1}(1+y^2)y^{-1}x^{-1} + q(1+y^2)x^{-1}(1+y^2)x^{-1}y^{-1}xy^{-1}x^{-1} + q(1+y^2)x^{-1}(1+y^2)x^{-1}(1+y^2)y^{-1}x^{-1}(1+y^2)x^{-1}(1+y^2)y^{-1}x^{-1},$$

where a factor of  $1+y^2$  indicates an edge which may be either included in or excluded from the corresponding admissible collection of labeled/colored subpaths. We present several examples for  $r_1 r_2 = 5$ , enumerating all admissible collections with their monomials, at [http://pages.uoregon.edu/drupel/dyck\\_examples.pdf](http://pages.uoregon.edu/drupel/dyck_examples.pdf).

*Proof of Theorem 6:* We divide the proof into a series of lemmas. First we make the following definitions.

**Definition 8.** Define the set  $\tilde{\mathcal{F}}(D_n)$  of non-overlapping collections  $\beta \subset \mathcal{P}(D_n)$  subject to conditions C1 and C2. Define  $\mathcal{T}^{\geq u}(D_n) \subset \tilde{\mathcal{F}}(D_n)$  to consist of those  $\beta$  satisfying the following condition only for  $m \geq u$ :

- C3<sup>op</sup>: there exists integers  $i, k, w, m$  such that  $\alpha(i, k) \in \beta$  is an  $(m, w)$ -Dyck suffix and none of the preceding  $U_{m-1,1} - wU_{m-2-\epsilon_{m-1},1}$  non-diagonal edges of  $\nu_i$  are supported on  $\beta$ .

**Lemma 9.** *If  $m \geq n-3$ , there do not exist  $i, w$  ( $1 \leq w < r_{m-\epsilon_{m-1}} - \delta_m$ ) so that  $\min\{t : i < t \leq U_{n-4,2}, s_{i,t} > s\}$  is of the form  $i + U_{m,2} - wU_{m-1-\epsilon_{m-1},2}$ . In particular, for any  $n \geq 2$ , the set  $\mathcal{T}^{\geq n-3}(D_n)$  is empty.*

*Proof.* We assume  $\epsilon_{m-1} = 0$ ; the case  $\epsilon_{m-1} > 0$  follows from this one. Since  $w < r_m - 1 - \epsilon_m$ , we have

$$U_{m,2} - wU_{m-1,2} \geq U_{m,2} - r_m U_{m-1,2} + (2 + \epsilon_m)U_{m-1,2} = (2 + \epsilon_m)U_{m-1,2} - U_{m-2,2} \geq U_{m-k,2}, \text{ for } k \geq 1.$$

Now if  $m \geq n-3$  and  $\tau := \min\{t : i < t \leq U_{n-4,2}, s_{i,t} > s\} = i + U_{m,2} - wU_{m-1,2}$ , then  $\tau \geq i + U_{n-4,2}$ . But this contradicts  $\nu_{U_{n-4,2}}$  being the highest labeled vertex in  $D_n$ .  $\square$

Let  $z_0 = x_0 = x$  and for  $n \geq 2$  write  $z_{n-1} = \sum_{\beta \in \mathcal{F}(D_n)} q \prod_{i=1}^{U_n} \beta_{[i]}$ . For each integer  $\ell$  we will use a parenthesized exponent  $(\ell)$  to denote a quantity with each  $r_k$  replaced by  $r_{k+\ell}$ .

**Lemma 10.** *Suppose  $n \geq 2$ . Then  $z_n^{(1)} = F_{r_2}(z_{n-1}) + \sum_{\beta \in \mathcal{T}^{\geq 1}(D_{n+1}^{(1)}) \setminus \mathcal{T}^{\geq 2}(D_{n+1}^{(1)})} q \prod_{i=1}^{U_{n+1}^{(1)}} \beta_{[i]}$ .*

*Proof.* This follows from a study of how the  $(1 + y^{r_2})^{-1}$  terms cancel in  $F_{r_2}(z_{n-1})$ . In particular, we make the following observations. The sum of the weights of a colored hook and the corresponding full hook of uncolored edges gives rise to a Laurent monomial under  $F_{r_2}$ . An edge  $\alpha$  in the support of  $\beta$  gives rise to a colored hook of type 1, 2, or 3 corresponding to the edge  $\alpha$  being horizontal, vertical not followed by a diagonal, or vertical followed by a diagonal, respectively. A missing edge  $\alpha$  gives rise to all collections of uncolored edges in a hook of type 1, 2, or 3 corresponding to the edge  $\alpha$  being horizontal, vertical not followed by a diagonal, or vertical followed by a diagonal, respectively.

Now consider an uncolored hook with a missing horizontal edge, followed by  $d$  included horizontal edges, and then an included vertical edge. Under  $F_{r_2}$  the weight of this configuration gives rise to the weights of all collections of horizontal edges in a hook of type 1 with an included vertical edge followed by  $d$  colored hooks of type 1 and then a colored hook of type 2. The sum is accounting for the included vertical edge in this case.  $\square$

In the following Lemma we consider a  $D_3$  with its first  $D_2$  removed as a single vertical edge and for  $\epsilon_3 = 1$  we consider a  $D_4$  with its first  $D_2$  removed as a vertical edge followed by a  $(-1, 1)$ -diagonal edge.

**Lemma 11.**

- (1) *Suppose  $k - \epsilon_{k-1} \geq 5$ . Then the weight of a missing  $D_{k-2}$  with its first  $D_{k-3-\epsilon_{k-3}}$  removed followed by a colored  $D_k$  simplifies to the weight of a colored  $D_{k-1-\epsilon_{k-1}}$ .*
- (2) *Suppose  $k - \epsilon_{k-1} \geq 5$ . Then the weight of a missing  $D_{k-2}$  followed by a colored  $D_{k-1-\epsilon_{k-1}}$  simplifies to the weight of a missing  $D_{k-2}$  with its first  $D_{k-3-\epsilon_{k-3}}$  removed.*
- (3) *Suppose  $m - \delta_m \geq 0$ . Then for  $1 \leq w < r_{m-\epsilon_{m-1}} - \delta_m$ , the weight of an  $(m, w)$ -Dyck suffix preceded by  $U_{m-1,1} - wU_{m-2-\epsilon_{m-1},1}$  missing non-diagonal edges is equal to the weight of an  $(m, w+1)$ -Dyck suffix preceded by  $U_{m-1,1} - (w+1)U_{m-2-\epsilon_{m-1},1}$  missing non-diagonal edges.*

*Proof.* Parts (1) and (2) follow from a simultaneous induction using Lemma 3 in the induction step. Part (3) follows from (1), (2), and Lemma 3.  $\square$

**Corollary 12.** *Suppose  $m - \delta_m \geq 0$ . Then for  $1 \leq w < r_{m-\epsilon_{m-1}} - \delta_m$ , the weight of an  $(m, w)$ -Dyck suffix preceded by  $U_{m-1,1} - wU_{m-2-\epsilon_{m-1},1}$  missing non-diagonal edges is equal to  $q^{-1}$ .*

*Proof.* We work by induction, the case  $m - \delta_m = 0$  is easy to check by hand. It follows from Lemma 3 that the hook sequences of an  $(m, r_{m-\epsilon_{m-1}} - \delta_m)$ -Dyck suffix and an  $(m-1, 1)$ -Dyck suffix are the same. Then one easily checks that  $U_{m-1,1} - (r_{m-\epsilon_{m-1}} - \delta_m)U_{m-2-\epsilon_{m-1},1} = U_{m-2,1} - U_{m-3-\epsilon_{m-2},1}$ , the case  $\epsilon_{m-1} > 0$  following from the case  $\epsilon_{m-1} = 0$ . The result now follows by induction using Lemma 11.3.  $\square$

We remind that a parenthesized exponent  $(\ell)$  denotes a quantity with each  $r_k$  replaced by  $r_{k+\ell}$ . In particular, note that  $F_{r_2}(x_k) = x_{k+1}^{(1)}$ .

**Lemma 13.** *Let  $u \geq 1$  and  $n \geq u + 4$ . Then*

$$F_{r_2} \left( \sum_{\beta \in \mathcal{T}^{\geq u}(D_n) \setminus \mathcal{T}^{\geq u+1}(D_n)} q \prod_{i=1}^{U_n} \beta_{[i]} \right) = \sum_{\beta \in \mathcal{T}^{\geq u+1}(D_{n+1}^{(1)}) \setminus \mathcal{T}^{\geq u+2}(D_{n+1}^{(1)})} q \prod_{i=1}^{U_{n+1}^{(1)}} \beta_{[i]}.$$

*Proof.* The proof follows by simultaneous induction with Lemma 14. We will assume  $n = u + 4$ , the case  $n > u + 4$  follows from this one using a similar argument. Also we restrict to the case  $\epsilon_{n-1} = 0$ , the case  $\epsilon_{n-1} > 0$  follows by a similar argument.

From Lemma 3, we can see that  $D_n$  begins with  $w$  copies of  $D_{n-1}$ ,  $1 \leq w < r_n - 1 - \epsilon_n$ , and the vertex  $\nu_{wU_{n-5,2}}$  is the ending vertex of the last  $D_{n-1}$ . Now  $\alpha(wU_{n-5,2}, U_{n-4,2})$  is the only  $(n-4, w)$ -Dyck suffix of  $D_n$  and so  $\beta \in \mathcal{T}^{\geq n-4}(D_n)$  implies  $\alpha(wU_{n-5,2}, U_{n-4,2}) \in \beta$  and none of the preceding  $U_{n-5,1} - wU_{n-6,1}$  non-diagonal edges are contained in  $\beta$ . Note that  $wU_{n-4,1} - U_{n-5,1} + wU_{n-6,1} = r_2wU_{n-5,2} - U_{n-5,1}$  and so the lowest vertex of these missing edges is  $\omega_{r_2wU_{n-5,2}-U_{n-5,1}}$ . Then Lemma 3 implies the subpath of  $D_n$  from  $\omega_0$  to  $\omega_{r_2wU_{n-5,2}-U_{n-5,1}}$  consists of  $w-1$  copies of  $D_{n-1}$ , followed by  $r_{n-1}-1$  copies of  $D_{n-2}$ , and then  $w-1$  copies of  $D_{n-3}$ . We will define  $j_i$  for  $0 \leq i \leq 2w + r_{n-1} - 3$  so that the  $\nu_{j_i}$  are the endpoints of these copies. Any subpath  $\alpha(i, k)$  can be decomposed as  $\alpha(i, j_e), \alpha(j_e, j_{e+1}), \dots, \alpha(j_{e+\ell}, k)$  where all but the first are Dyck prefixes. It is easy to see that  $\alpha(i, j_e)$  has the same label/color as  $\alpha(i, k)$  and if  $\alpha(i, k)$  was an  $(m, w')$ -Dyck suffix then so is  $\alpha(i, j_e)$ .

Combining the above considerations we see that  $\sum_{\beta \in \mathcal{T}^{\geq u}(D_n) \setminus \mathcal{T}^{\geq u+1}(D_n)} q \prod_{i=1}^{U_n} \beta_{[i]}$  can be rewritten as:

$$\begin{aligned} & \sum_{w=1}^{r_n-2-\epsilon_n} q \left( \sum_{\beta \in \mathcal{F}(D_{n-1})} \prod_{i=1}^{U_{n-1}} \beta_{[i]} \right)^{w-1} \left( \sum_{\beta \in \mathcal{F}(D_{n-2})} \prod_{i=1}^{U_{n-2}} \beta_{[i]} \right)^{r_{n-1}-1} \left( \sum_{\beta \in \mathcal{F}(D_{n-3})} \prod_{i=1}^{U_{n-3}} \beta_{[i]} \right)^{w-1} q^{-1} \\ &= \sum_{w=1}^{r_n-2-\epsilon_n} q (q^{-1}x_{n-2})^{w-1} (q^{-1}x_{n-3})^{r_{n-1}-1} (q^{-1}x_{n-4})^{w-1} q^{-1}, \end{aligned}$$

where the equality follows from Lemma 14. Applying  $F_{r_2}$  and noting that  $F_{r_2}(q) = q$  completes the proof.  $\square$

**Lemma 14.** *Suppose  $n \geq 3$ . Then*

$$(1) \quad x_{n-1} = z_{n-1} - \sum_{m=5}^n \underbrace{F_{r_1} F_{r_2} F_{r_1} \cdots}_{n-m} \left( \sum_{\beta \in \mathcal{T}^{\geq 1}(D_m^{(m-n)}) \setminus \mathcal{T}^{\geq 2}(D_m^{(m-n)})} q \prod_{i=1}^{U_m^{(m-n)}} \beta_{[i]} \right) = \sum_{\beta \in \mathcal{F}(D_n)} q \prod_{i=1}^{U_n} \beta_{[i]}.$$

*Proof.* This follows from simultaneous induction with Lemma 13 as in the proof of [3, Lemma 20].  $\square$

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